Inductive characterizations of hyperquadrics

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INDUCTIVE CHARACTERIZATIONS OF HYPERQUADRICS

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Abstract. We give two characterizations of hyperquadrics: one as non-degenerate smooth projective varieties swept out by large dimensional quadric subvarieties passing through a point; the other as $LQEL$-manifolds with large secant defects.

1. Introduction

We work over an algebraically closed field of characteristic zero. In [Ein], Ein proved that if X is an n-dimensional smooth projective variety containing an m-plane Π_0 whose normal bundle is trivial, with $m \geq n/2 + 1$, then there exists a smooth projective variety Y and a vector bundle E over Y such that $X \simeq \mathbb{P}(E)$ and Π_0 is a fiber of $X \to Y$. The bound on m was improved to $m \geq n/2$ by Wiśniewski in [Wis]. Later on, Sato [Sat] studied projective smooth n-folds X swept out by m-dimensional linear subspaces, i.e. through every point of X, there passes through an m-dimensional linear subspace. If $m \geq n/2$, he proved that either X is a projective bundle as above or $m = n/2$. In the latter case, X is either a smooth hyperquadric or the Grassmanian variety parametrizing lines in $\mathbb{P}^{m+1}$.

A natural problem is to extend these results to the case where linear subspaces are replaced by quadric hypersurfaces. In this paper, we will consider a smooth projective non-degenerate variety $X \subsetneq \mathbb{P}^N$ of dimension n, which is swept out by m-dimensional irreducible hyperquadrics passing through a point (for the precise definition see section 3). Examples of such varieties include Severi varieties (see [Zak]), or more generally LQEL manifolds of positive secant defect(see section 2 below). As it turns out, the number m is closely related to the secant defect of X, which makes it hard to construct examples with big m.

Our main theorem is to show (cf. Thm. 2) that if $m > [n/2] + 1$, then $N = n+1$ and X is itself a hyperquadric. This gives a substantial improvement to the Main theorem 0.2 of [KS], where the same claim is proved under the assumption that a general hyperquadric in the family is smooth and that $m \geq 3n/5 + 1$. Our proof here, based on ideas contained in [IR2] and [Rus], is much simpler and is completely different from that in [KS]. However, we should point out that a more general result, without assuming the quadric subspaces pass all through a fixed point, is proven in [KS].

The same idea of proof, combined with the Divisibility Theorem of [Rus], allows us to prove (cf. Corollary 3) that for an n-dimensional $LQEL$-manifold, either it is a hyperquadric or its secant defect is no bigger than $\frac{n+8}{3}$. This improves Corollary 0.11, 0.14 of [KS]. It also gives positive support to the general believing that hyperquadrics are the only LQEL manifolds with large secant defects.

2. Preliminaries

Let $\delta = \delta(X) = 2n + 1 - \dim(SX)$ be the *secant defect* of a non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$, where

$$SX = \overline{\bigcup_{\substack{x \neq y \\ x,y \in X}} \langle x, y \rangle} \subseteq \mathbb{P}^N$$

is the *secant variety* of $X \subset \mathbb{P}^N$.

Recall([KS], [IR1]) that a smooth irreducible non-degenerate projective variety $Z \subset \mathbb{P}^N$ is said to be *conically connected* (CC for short) if through two general points there passes an irreducible conic contained in Z. Such varieties have been studied and classified in [IR1] and [IR2].

We begin with a simple but very useful remark, which is probably well known but we were not able to find a reference.

Lemma 1. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and let $z \in X$ be a point. If there exists a family of smooth rational curves of degree d on X passing through z and covering X, then through two general points $x, y \in X$ there passes such a curve.*

In particular, if $d = 1$, then $X \subset \mathbb{P}^N$ is a linearly embedded $\mathbb{P}^n$. If $d = 2$ and if $X \subset \mathbb{P}^N$ is non-degenerate, then $X \subset \mathbb{P}^N$ is conically connected.

Proof. By Theorem II.3.11 [Kol], there exists finitely many closed subvarieties (depending on z) $V_i \subsetneq X$, $i = 1, \ldots, l$, such that for any nonconstant morphism $f : \mathbb{P}^1 \to X$ with $f(0) = z$, $\deg(f_*(\mathbb{P}^1)) = d$ and with $f(\mathbb{P}^1) \nsubseteq \cup_{i=1}^l V_i$, we have f^*T_X is ample. Now take a general point $x \in X \setminus \cup_{i=1}^l V_i$ and a smooth rational curve $C \subset X$ of degree d passing through x and z. The above result implies that $f^*T_X = T_X|_C$ is ample and hence that $N_{C|X}$ is ample. Thus there exists a unique irreducible component W_x of the Hilbert schemes of rational curves of degree d contained in X and passing through x containing $[C]$. Since

$N_{C|X}$ is ample, it is well known that deformations of C parametrized by W_x cover X. Therefore given a general point $y \in X$, we can find a smooth rational curve of degree d contained in X joining x and y, proving the first part of the assertion. To conclude the proof it is sufficient to recall that linear subspaces of $\mathbb{P}^N$ are the unique irreducible subvarieties containing the line through two general points of itself. $\square$

The following general result on CC-manifolds is proved in [IR2].

Proposition 1. ([IR2, Prop. 3.2]) *Let $X \subset \mathbb{P}^N$ be a CC-manifold and let $C = C_{x,y}$ be a general conic through the general points $x, y \in X$. Then*

$$n + \delta(X) \geq -K_X \cdot C \geq n + 1.$$

If moreover $\delta(X) \geq 3$, then $X \subset \mathbb{P}^N$ is a Fano manifold with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$*, whose index $i(X)$ satisfies*

$$\frac{n + \delta(X)}{2} \geq i(X) \geq \frac{n+1}{2}.$$

Now consider a smooth projective variety $X \subset \mathbb{P}^N$. For a general point $x \in X$, let Y_x be the Hilbert scheme of lines on $X \subset \mathbb{P}^N$ passing through x, which can be naturally regarded as a sub-variety in $\mathbb{P}((t_x X)^*) = \mathbb{P}^{n-1}$, where $t_x X$ is the affine tangent space to X at x. The variety Y_x is the first instance of the so-called *variety of minimal rational tangents*, introduced and extensively studied by Hwang and Mok (see [Hwa] and the references therein). When $X \subset \mathbb{P}^N$ is a Fano manifold with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}(1) \rangle$, there exists a deep connection between geometrical properties of $Y_x \subset \mathbb{P}^{n-1}$ and the index of X. The following result contained in [IR1, Prop. 2.4] is essentially due to Hwang and Kebekus, cf. [HK, Th. 3.14].

Proposition 2. ([HK, Th. 3.14] and [IR1, Prop. 2.4]) *Let $X \subset \mathbb{P}^N$ be a Fano manifold with* $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H \rangle$ *and $-K_X = i(X)H$, H being the hyperplane section and $i(X)$ the index of X.*

(i) *If $i(X) > \frac{n+1}{2}$, then $X \subset \mathbb{P}^N$ is ruled by lines and for general $x \in X$ the Hilbert scheme of lines through x, $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*) = \mathbb{P}^{n-1}$, is smooth. If $i(X) \geq \frac{n+3}{2}$, Y_x is also irreducible.*

(ii) *If $i(X) \geq \frac{n+3}{2}$ and $SY_x = \mathbb{P}^{n-1}$, then $X \subset \mathbb{P}^N$ is a CC-manifold.*

(iii) *If $i(X) > \frac{2n}{3}$, then $X \subset \mathbb{P}^N$ is a CC-manifold with $\delta(X) > \frac{n}{3}$ and such that $SY_x = \mathbb{P}^{n-1}$.*

Recall that (cf. [KS], [Rus], [IR2]) a smooth irreducible non-degenerate variety $X \subset \mathbb{P}^N$ is said to be a *local quadratic entry locus manifold of type* $\delta \geq 0$ (*LQEL*-manifold for short) if for general $x, y \in X$ distinct points, there exists a hyperquadric of dimension $\delta = \delta(X)$ contained in X and passing through x, y. By definition, a LQEL manifold of positive secant defect is conically connected, but the converse is not true. For example, a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geq 3$ is conically connected but not a LQEL-manifold. Severi varieties and Scorza varieties are basic examples of LQEL manifolds ([Zak]).

A systematic study of LQEL manifolds has been succesively carried out by Russo in [Rus], in particular, the following remarkable theorem has been proved in [Rus].

Theorem 1. ([Rus, Th. 2.8]) *For an n-dimensional LQEL-manifold $X \subset \mathbb{P}^N$ of type $\delta \geq 3$, let $x \in X$ be a general point and let $Y_x \subset \mathbb{P}^{n-1}$ be the Hilbert scheme of lines on X passing through x. Then $Y_x \subset \mathbb{P}^{n-1}$ is a LQEL-manifold of type $\delta - 2$, of dimension $(n+\delta)/2 - 2$ and such that $SY_x = \mathbb{P}^{n-1}$. Let $\delta = 2r_X + 1$, or $\delta = 2r_X + 2$. Then 2^{r_X} divides $n - \delta$.*

3. Varieties swept out by hyperquadrics

Through out this section, let $X \subsetneq \mathbb{P}^N$ be an n-dimensional non-degenerate projective smooth variety which satisfies the following two conditions:

i) through a general point $x \in X$, there passes an irreducible reduced m-dimensional quadric $Q_x \subset X \subset \mathbb{P}^N$, where m is a fixed natural number (i.e. the linear span $< Q_x >$ of Q_x in $\mathbb{P}^N$ is a linear subspace of dimension $m + 1$ and $Q_x \subset < Q_x >$ is a quadric hypersurface);

ii) there exists a point $z \in X$ such that for $x \in X$ general, the quadric Q_x passes through z.

We will say such a variety is *swept out by m-dimensional hyperquadrics passing through* $z \in X$. For example, a LQEL manifold with secant defect $\delta > 0$ is swept out by δ-dimensional hyperquadrics passing through a point. By Lemma 1, a smooth variety is conically connected if and only if it is swept out by a 1-dimensional hyperquadrics passing through a point.

Lemma 2. *The secant defect δ of a variety $X \subset \mathbb{P}^N$ swept out by m-dimensional hyperquadrics passing through a point $z \in X$ satisfies $\delta \geq m$.*

Proof. Let $\mathrm{Hilb}^{conic,z}(X)$ be the Hilbert scheme of conics in X passing through z and let $W_1, \cdots, W_k$ be its irreducible components. If z is a singular point of Q_x for $x \in X$ general, then the line $< z, x >$ would be contained in X and by Lemma 1 $X \subset \mathbb{P}^N$ would be degenerated contrary to our assumption. Thus for general $x \in X$, z is a smooth point of Q_x and the Hilbert scheme $\mathrm{Hilb}^{conic,z}(Q_x)$ is irreducible, so that there exists some $i \in \{1, \cdots, k\}$ such that $\mathrm{Hilb}^{conic,z}(Q_x) \subset W_i$. This implies that there exists a component $W := W_{i_0}$ containing $\mathrm{Hilb}^{conic,z}(Q_x)$ for $x \in X$ general. This gives the dimension estimate:

$$\dim W \geq n + m - 2. \tag{3.1}$$

Reasoning as in the proof of Lemma 1, if we take a general point $x \in X$ and an irreducible conic $[C] \subset Q_x$ joining x and z, then we can suppose that $N_{C|X}$ is ample. Thus W is smooth at the point $[C]$ and

$$\dim(W) = \dim H^0(C, N_{C|X} \otimes \mathcal{O}_C(-z)) = -K_X \cdot C - 2.$$

Combining with (3.1), we obtain

$$-K_X \cdot C \geq n + m. \tag{3.2}$$

By Lemma 1, $X \subset \mathbb{P}^N$ is conically connected so that Proposition 1 gives $n + \delta \geq -K_X \cdot C \geq n + m$, yielding $\delta \geq m$. □

An immediate consequence of this lemma and Prop. 1 is the following result.

Corollary 1. *If $m \geq 3$, then $X \subset \mathbb{P}^N$ is a Fano variety with* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}(1) \rangle$ *and the index $i(X)$ satisfies*

$$\frac{n+\delta}{2} \geq i(X) \geq \frac{n+m}{2}.$$

Recall that for a general point $x \in X$, the variety Y_x is the Hilbert scheme of lines on X passing through x.

Lemma 3. *Assume that $m \geq 3$. Then Y_x is smooth irreducible of dimension $i(X) - 2$. If moreover $m > n/3$, then Y_x is non-degenerate and $SY_x = \mathbb{P}^{n-1}$.*

Proof. Corollary 1 yields $i(X) \geq (n+m)/2 \geq (n+3)/2$. By part (i) of Prop. 2 we deduce that $Y_x \subset \mathbb{P}^{n-1}$ is not empty and irreducible. If l_x is a line through x, then $\dim(Y_x) = H^0(N_{l_x|X}) = -K_X \cdot l_x - 2 = i(X) - 2$. The last part follows form (iii) of Prop. 2. □

In the sequel we shall use the following simple remark.

Lemma 4. *Assume $n \geq 2$ and $\delta \geq 1$. If $Y_x \subset \mathbb{P}^{n-1}$ is a non-degenerate hypersurface, then $Y_x \subset \mathbb{P}^{n-1}$ is a smooth quadric hypersurface and $X \subset \mathbb{P}^{n+1}$ is a smooth quadric hypersurface.*

Proof. Since $\delta \geq 1$, the second fundamental form $|II_{x,X}| \subseteq |\mathcal{O}_{\mathbb{P}^{n-1}}(2)|$ is a linear system of quadrics of dimension $N - n - 1$ (see for example [Rus, Thm. 2.3 (1)]). Since $Y_x \subset \mathbb{P}^{n-1}$ is contained in the base locus scheme of $|II_{x,X}|$ and since it is a non-degenerate hypersurface, we obatain that $Y_x \subset \mathbb{P}^{n-1}$ is a quadric hypersurface and that $N = n + 1$, i.e. $X \subset \mathbb{P}^{n+1}$ is a hypersurface. Let $l_x \subset X$ be a line passing through x. Reasoning as in the proof of Lemma 3 we get, by adjunction,

$$n - 2 = \dim(Y_x) = -K_X \cdot l_x - 2 = -(\deg(X) - n - 2) - 2,$$

that is $\deg(X) = 2$ as claimed. □

We now prove a substantial improvement of the Main Theorem 0.2 of [KS], where the same claim is proved under the stronger assumption that a general hyperquadric is smooth and that $m \geq 3n/5+1$ if $n = 5, 6$ or 10 and $m \geq 3n/5$ otherwise.

Theorem 2. *Let $X^n \subsetneq \mathbb{P}^N$ be a smooth non-degenerate variety, which is swept out by m-dimensional hyperquadrics passing through a point. If $m > [n/2]+1$, then $N = n+1$ and X is itself a smooth hyperquadric.*

Proof. The condition $m > [n/2] + 1$ implies $m \geq 3$. By Lemma 3 we know that $Y_x \subset \mathbb{P}^{n-1}$ is a smooth non-degenerate variety. Reasoning as in the proof of Lemma 2 we can suppose that, for $x \in X$ general, z is a smooth point of Q_x, so that lines on the quadric Q_x passing through z are parameterized by an $(m-2)$-dimensional quadric hypersurface $\tilde{Q}_x \subset \mathbb{P}^{n-1}$. Clearly $\tilde{Q}_x \subset Y_x$. By assumption, $m - 2 > [(n-2)/2]$, so $Y_x \subset \mathbb{P}^{n-1}$ contains a high dimensional variety which is a hypersurface in its linear span in $\mathbb{P}^{n-1}$. Then [Zak, Corollary I.2.20] implies that $Y_x \subset \mathbb{P}^{n-1}$ is itself a hypersurface and the conclusion now follows from Lemma 4. □

The following corollary is analogue to results in [Ein], [Wis] and [Sat], where they considered linear subspaces instead of quadric subvarieties.

Corollary 2. *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate variety of dimension n and $Q \subset X$ a smooth quadric subvariety of dimension m whose normal bundle $N_{Q|X}$ is isomorphic to $\mathcal{O}_Q(1)^{\oplus n-m}$. If $m > [n/2] + 1$, then X is a hyperquadric.*

Proof. Let $\mathcal{I}_q$ be the ideal sheaf of a point $q \in Q$. By the exact sequence $0 \to N_{Q|X} \otimes \mathcal{I}_q \to N_{Q|X} \to N_{Q|X,q} \to 0$, we get $H^1(Q, N_{Q|X} \otimes \mathcal{I}_q) = 0$, since $N_{Q|X} \simeq \mathcal{O}_Q(1)^{\oplus n-m}$ is globally generated and $H^1(Q, \mathcal{O}_Q(1)) = 0$. Similarly, since T_Q is globally generated and $H^1(Q, T_Q) = 0$, we obtain $H^1(Q, T_Q \otimes \mathcal{I}_q) = 0$. Note that the following sequence is exact:

$$0 \to T_Q \otimes \mathcal{I}_q \to T_X|_Q \otimes \mathcal{I}_q \to N_{Q|X} \otimes \mathcal{I}_q \to 0.$$

The long exact sequence of cohomology gives $H^1(Q, T_X|_Q \otimes \mathcal{I}_q) = 0$ and the following sequence is exact:

(3.3)

$$0 \to H^0(Q, T_Q \otimes \mathcal{I}_q) \to H^0(Q, T_X|_Q \otimes \mathcal{I}_q) \to H^0(Q, N_{Q|X} \otimes \mathcal{I}_q) \to 0.$$

Let $\mathrm{Mor}(Q, X; q)$ be the variety parameterizing morphisms from Q to X fixing the point q. Then it is smooth at $\iota : Q \to X$, the natural inclusion.

Consider the evaluation map $ev : Q \times \mathrm{Mor}(Q, X; q) \to X$. Take a point $p \in Q - \{q\}$. The tangent map to ev at point (p, ι) is

$$T_pQ \oplus H^0(Q, T_X|_Q \otimes \mathcal{I}_q) \to T_{p,X}$$

given by

$$(u, \sigma) \mapsto T_p\iota(u) + \sigma(p) = u + \sigma(p).$$

Thus the image contains T_pQ. To show it is surjective, we just need to show that the composition map $H^0(Q, T_X|_Q \otimes \mathcal{I}_q) \to T_{p,X} \to N_{p,Q|X}, \sigma \mapsto [\sigma(p)]$ is surjective. By the exact sequence (3.3), it is enough to show that $H^0(Q, N_{Q|X} \otimes \mathcal{I}_q) \to N_{p,Q|X}$ is surjective, i.e. $H^0(Q, \mathcal{O}_Q(1) \otimes \mathcal{I}_q) \to k_p$ is surjective. This is immediate from the very ampleness of $\mathcal{O}_Q(1)$.

In particular, this implies that the map ev is smooth at points (p, ι) for $p \neq q$. Thus the deformations of Q while fixing q dominant X. Now we can apply the precedent theorem to conclude. □

Next we will consider the case $m = [n/2] + 1$ with $n \geq 3$.

Proposition 3. *If $N \geq 3n/2$ and $m = [n/2] + 1$ with $n \geq 3$, then X is projectively isomorphic to one of the following:*

i) the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;

ii) the Plücker embedding $\mathbb{G}(1, 4) \subset \mathbb{P}^9$;

iii) the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$;

iv) a general hyperplane section of ii) or iii).

Proof. As $\delta \geq m > n/2$, by Zak's linear normality theorem ([Zak]), we have $SX = \mathbb{P}^N$. Thus $\delta = 2n + 1 - N \geq [n/2] + 1$, which gives that $N \leq 2n - [n/2]$. By hypothesis, $N \geq 3n/2$, thus $N = 2n - [n/2]$, which gives $\delta = m$. As a consequence, $-K_X \cdot C = n + \delta$ for a generic conic C, which implies that X is a LQEL manifold of type m by [IR2, Prop. 3.2]. Now the claim is given by the classification result in [Rus, Cor. 3.1]. □

Remark 1. Here we give an outline of an approach to classify such varieties X with $m = [n/2] + 1$, based on Hartshorne's conjecture. We may assume that Y_x is not a hypersurface in $\mathbb{P}^{n-1}$, i. e. $n-2-\dim Y_x \geq 1$. By the proof of Prop. I.2.16 [Zak], for any hyperplane $H \subset \mathbb{P}^{n-1}$ containing the linear span of $\tilde{Q}_x$ and T_yY for some $y \in Y$, H is tangent to Y_x along some variety $Z \subset \tilde{Q}_x$. The dimension of Z is bounded by

$$n - 2 - \dim Y_x \geq \dim Z \geq 2(m-2) - \dim Y_x = 2[n/2] - 2 - \dim Y_x.$$

Consider the Gauss map: $\gamma_{n-2} : \mathcal{P}_{n-2} \to (\mathbb{P}^{n-1})^*$ (cf. I.2 [Zak]). By definition, $\gamma_{n-2}^{-1}(H)$ contains the variety $Z \times \{H\}$. If $2n-2 < 3i(X)-6$, then Harsthorne's conjecture implies that $Y_x \subset \mathbb{P}^{n-1}$ is a complete intersection. By Prop. I. 2.10 [Zak], the map γ_{n-2} is finite, which gives $\dim Z = 0$. We deduce that n is odd and $i(X) = n - 1$, so X is a smooth Del Pezzo varieties, which have been completely classified.

Thus we may assume $2n - 2 \geq 3i(X) - 6$, which gives $2n + 4 \geq 3/2(n + m) = 3/2(n + [n/2] + 1)$. This implies that $n \leq 11$ or $n = 13$. When $n \leq 11$, we obtain that $i(X) \geq n - 2$, thus X is a Fano variety with $Pic \simeq \mathbb{Z}$ and of coindex at most 3, i. e. X is either a Del Pezzo variety or a Mukai variety. The case $n = 13$ with $i(X) = 10$ requires a more detailed study.

4. LQEL-manifolds with large secant defects

The idea contained in the proof of Theorem 2 can be combined with the Divisibility Theorem of [Rus], obtaining new constraints for the existence of $LQEL$-manifold with large secant defects.

Let $X \subset \mathbb{P}^N$ be a $LQEL$-manifold of type $\delta \geq 2k + 1$. We define inductively a sequence of smooth varieties: $Y_1 := Y_x \subset \mathbb{P}^{n-1}$ and let $Y_{j+1} \subset \mathbb{P}^{\dim(Y_j)-1}$ be the Hilbert scheme of lines on Y_j passing through a general point of it, for $k - 1 \geq j \geq 1$. By the previous theorem, we know that $Y_j \subset \mathbb{P}^{\dim(Y_{j-1})-1}$ is a $LQEL$-manifold of type $\delta - 2j$ with $SY_j = \mathbb{P}^{\dim(Y_{j-1})-1}$. Furthermore for $j \leq k-1$, $Y_j \subset \mathbb{P}^{\dim(Y_{j-1})-1}$ is a Fano variety with $\mathrm{Pic}(Y_j) = \mathbb{Z}\langle\mathcal{O}(1)\rangle$ ([Rus]). Let i_j be the index of Y_j and $i_0 = (n + \delta)/2$ the index of X. The following lemma can also be deduced from the Divisibility Theorem cited above.

Lemma 5.
$$i_j = \frac{n-\delta}{2^{j+1}} + \delta - 2j, \ 0 \leq j \leq k-1.$$

Proof. By Theorem 1, we have

$$2i_j = \dim Y_j + \delta(Y_j) = i_{j-1} - 2 + \delta - 2j,$$

which gives $2(i_j + 2j - \delta) = i_{j-1} + 2(j-1) - \delta$. We deduce that $i_j + 2j - \delta = (i_0 - \delta)/2^j$, concluding the proof. □

Theorem 3. *Let $X \subset \mathbb{P}^N$ be an n-dimensional LQEL-manifold of type δ. If*

$$\delta > 2[\log_2 n] + 2 \text{ or } \delta > \min_{k\in\mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\},$$

then $N = n+1$ and $X \subset \mathbb{P}^{n+1}$ is a quadric hypersurface.

Proof. If $\delta > 2[\log_2 n] + 2$, then $n < 2^r$, where $r = [(\delta-1)/2]$. By Theorem 1, 2^r divides $n - \delta$. This is possible only if $\delta = n$. Thus X is a hyperquadric. Now assume we have the second inequality. Note that for a fixed n, the minimum $\min_{k\in\mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\}$ is achieved for some $k \leq n/2$, so we may assume that for some $k \leq n/2$, we have $\delta > \frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1} = 2k + \frac{n-2k}{2^{k-1}+1} \geq 2k$, so that $\delta \geq 2k+1$. Now we can consider the variety $Y_k \subset \mathbb{P}^{\dim Y_{k-1} - 1}$. Note that $\dim Y_k = i(Y_{k-1}) - 2$ and

$$\dim Y_{k-1} = 2i_{k-1} - \delta(Y_{k-1}) = \frac{n-\delta}{2^{k-1}} + \delta - 2k + 2.$$

On the other hand, $Y_k \subset \mathbb{P}^{\dim(Y_{k-1})-1}$ is non-degenerate and it contains a hyperquadric of dimension $\delta - 2k$, which is strictly bigger than $(\dim Y_{k-1} - 2)/2$ under our assumption on δ. Now [Zak, Corollary I.2.20] implies that $Y_k \subset \mathbb{P}^{\dim(Y_{k-1})-1}$ is a hypersurface. Since it is a non-degenerate hypersurface by Theorem 1, a repeated application of Lemma 4 yields the conclusion. □

We now state a sharper Linearly Normality Bound for $LQEL$-manifolds, see [Zak, II.2.17]. Moreover, in [Rus, Cor. 3.1, Cor. 3.2] Russo has classified n-dimensional $LQEL$-manifolds of type $\delta \geq n/2$. Combining these results with the bound on δ in the Theorem 3 we are able to classify the extremas cases of the bounds.

Corollary 3. *Let $X \subset \mathbb{P}^N$ be a LQEL-manifold of type δ, not a quadric hypersurface. Then*

$$\delta \leq \min_{k\in\mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\} \leq \frac{n+8}{3}$$

and

$$N \geq \dim(SX) \geq 2n+1-\min_{k\in\mathbb{N}}\{\frac{n}{2^{k-1}+1}+\frac{2^k k}{2^{k-1}+1}\} \geq \frac{5}{3}(n-1).$$

Furthermore $\delta = \frac{n+8}{3}$ *if and only if* $X \subset \mathbb{P}^N$ *is projectively equivalent to one of the following:*

i) *a smooth 4-dimensional quadric hypersurface* $X \subset \mathbb{P}^5$*;*
ii) *the* 10*-dimensional spinor variety* $S^{10} \subset \mathbb{P}^{15}$*;*
iii) *the* E_6*-variety* $X \subset \mathbb{P}^{26}$ *or one of its isomorphic projection in* $\mathbb{P}^{25}$*;*
iv) *a* 16*-dimensional linearly normal rational variety* $X \subset \mathbb{P}^{25}$*, which is a Fano variety of index* 12 *with* $SX = \mathbb{P}^{25}$*, dual defect* $\mathrm{def}(X) = 0$ *and such that the base locus scheme* $C_x \subset \mathbb{P}^{15}$ *of* $|II_{x,X}|$ *is the union of* 10*-dimensional spinor variety* $S^{10} \subset \mathbb{P}^{15}$ *with* $C_pS^{10} \simeq \mathbb{P}^7$*,* $p \in \mathbb{P}^{15} \setminus S^{10}$*.*

Proof. We shall prove only the second part. If $\delta = \frac{n+8}{3}$, then $n-\delta = \frac{2n-8}{3}$. Suppose $\delta = 2r_X+1$, so that $n-\delta = \frac{12r_X-18}{3}$. By Theorem 1 we deduce that 2^{r_X} should divide $4r_X-6$, which is not possible.

Suppose now $\delta = 2r_X+2$, so that $n-\delta = \frac{12r_X-12}{3} = 4(r_X-1)$. Since 2^{r_X} has to divide $4(r_X-1)$, we get $r_X = 1,2,3$ and, respectively, $n = 4,10,16$ with $\delta = 4,6$, respectively 8. The conclusion follows from [Rus, Cor 3.1 and Cor. 3.2]. ☐

Let us observe that Lazarsfeld and Van de Ven posed the question if for an irreducible smooth projective non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$ with $SX \subsetneq \mathbb{P}^N$ the secant defect is bounded, see [LVdV]. This question was motivated by the fact that for the known examples we have $\delta(X) \leq 8$, the bound being attained for the sixteen dimensional Cartan variety $E_6 \subset \mathbb{P}^{26}$, which is a $LQEL$-variety of type $\delta = 8$. Based on these remarks and on the above results one could naturally formulate the following problem.

Question: Is a $LQEL$-manifold $X \subset \mathbb{P}^N$ with $\delta > 8$ a smooth quadric hypersurface?

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